Validating Shock Capturing Schemes On The DamBreak Problem

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Abstract

In this report we discuss some numerical techniques for approximating the Shallow-water equations. In particular finite difference schemes, adaptation of Roe's approximate Riemann solver and the HLL scheme of (Harten-Lax-Van Leer) with the objective of accurately approximating the solution of Shallow-water equations over variable topography. Some tests are presented as preliminary validation of the proposed framework. Satisfactory comparisons have been obtained. This report shows progress towards a more complete validation of the schemes and demonstrate a critical need to improve procedure if such progress is to be sustained.

Key words: St-Venant, Finite Difference, DamBreak Problem, Partial Differential Equations

1 Introduction

Considerable work has been devoted to the numerical solution of the *Shallow-Water Equations* (SWE) not only for their inherent importance as regards modeling of many physical processes, ranging from river and channels flows to estuarine circulation and floods due to the dam or dike failure, but also for their mathematical difficulties, namely its non-linearity can give rise to discontinuous solutions currently referred to as bores or jumps. In fact the SWE are constitute a nonlinear system of partial differential equations (in one and two dimensions) of the hyperbolic type with a nonlinear source term.

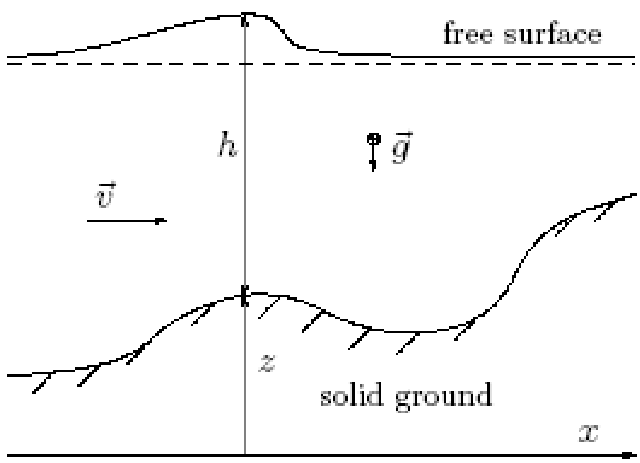

Figure 1: Schematic of the coordinates and variables of the shallow water model

We consider here one dimensional *shallow-water flow* and understand by such the movement of a layer of water as a whole over a variable bottom that can be described by specifying at any point and instant of time the thickness of the water layer or depth, h, and the average horizontal velocity of the fluid, v (see figure 1). This description implicitly assumes that no other than horizontal movement takes place, thus neglecting vertical accelerations and velocities of the fluid and the horizontal velocity $v(x, t)$ is roughly constant through any vertical cross section. This is true if we consider (fluid incompressible) small amplitude waves in a fluid that is shallow relative to the wave length. It can be extended to two dimensions by considering also *horizontal* movement in the transverse direction to the one considered above. Under these assumptions the equations that model 1-D open-channel flow (Saint-Venant equations), in a wide rectangular channel, can be written in a conservative or divergent form along the direction of movement

$$\mathbf{U}_t + \mathbf{F}_x + \mathbf{S} = 0 \tag{1}$$

where the vector $\mathbf{U}$ is the state variables vector, $\mathbf{F}$ flux of the state variables, $\mathbf{S}$ represents sinks and sources of the momentum arising from the bed slope and friction losses. They can be expressed in terms of flow variables as

$$\mathbf{U} = \begin{pmatrix} h \\ Q \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} Q \\ Q^2/h + gh^2/2 \end{pmatrix} \qquad \mathbf{S} = \begin{pmatrix} 0 \\ gh(S_{0x} - S_{fx}) \end{pmatrix} \tag{2}$$

where g is the gravitational acceleration, h is the water depth of flow, $Q = vh$ the discharge, $S_0 = -z_x$ is the bed slope expressed in term of the spatial derivative of the bottom elevation z, the friction lost S_{fx} can be evaluated from the Manning formula given by

$$S_{fx} = \frac{N^2\, v\, |v|}{h^{4/3}}$$

where N is the Manning coefficient.

In a non-divergent form it can be written as

$$\mathbf{U}_t + \mathbf{J}(\mathbf{U})\,\mathbf{U}_x = 0 \tag{3}$$

The Jacobian matrix of the flux is

$$\mathbf{J}(\mathbf{U}) = \frac{\partial \mathbf{F}}{\partial \mathbf{U}} = \begin{pmatrix} 0 & 1 \\ gh - v^2 & 2v \end{pmatrix} \tag{4}$$

with eigenvalues and eigenvectors

$$\lambda^{1,2} = v \pm c, \qquad \mathbf{e}^{1,2} = \begin{pmatrix} 1 \\ v \pm c \end{pmatrix} \qquad c = \sqrt{(gh)} \tag{5}$$

There are two distinct eigenvalues, these values are obviously real as long as h is positive, therefore the shallow-water equations are hyperbolic and exhibit wave like behavior(Smoller 1983). For hyperbolic systems such as (1) information travels from one point to another at speeds equal to the eigenvalues of the Jacobian matrix. For a more complete discussion of the properties of the hyperbolic systems see for example (Leveque 1990)

2 Motivation

There are a variety of numerical techniques for approximating the shallow-water equations. But in practical case difficulties arise from the bottom slope term which introduces new features in the system. As discussed by Nujic (1995), improper treatment of this term may lead to great inaccuracies if strong variations in the bottom topography are present. The core problem is well understood: over irregular topography, momentum-conserving algorithms tend to balance incorrectly the hydrostatic pressure contributions.

Some work have been done and proposed solutions to this problem. Among the most recent model worthy of note is Nujic (1995) scheme where he separate the pressure part from the convection part. He used the water level variable instead of the depth and he extracted the gravitational terms from flux functions. Then, using the Shu and Osher scheme (1988), he computed the flux vector and obtained accurate results for variable topography. We would like to consider the more general case (one-dimensional), by taking into account the slope, section of arbitrary width, friction and hydraulic jump on a irregular topography and a complex flow.

For this purpose, we will make use of a real case problem, a river located in the province of Quebec. The river reach is chosen for the complexity of the flow and for the quality of the field data. Two-dimensional simulation has been performed and favorable comparisons with laboratory measurements have been obtained (see Belanger, Carreau and Vincent (2000)). We will use these results for the sake of comparison. This study will serve as a strong validation case for the effect of the flood plain on numerical accuracy. Moreover, we hope to gain a better understanding of the physics of interaction between flood plain with the SWE over irregular topography.

In this first report, we present the schemes and validate them on the Dam-Break problem. Four explicit conservative schemes are examined for the solution of the flux convective part. These include naive finite difference scheme, first-order (Lax-Friedrichs), Nujic(1995) model, which is based on the "flux-splitting" technique with Essentially Non-Oscillatory (ENO) schemes, Harten, Lax and van Leer (HLL) solver (Minhgam and Causon, 1998) and a simplified version of Roe's approximate solver, belongs to shock capturing scheme. Second-order is achieved with flux limiters and a reconstruction of MUSCL type. Well known analytical solution to the Dam-Break problem is used to assess the performance of these shock-capturing schemes. Much of the mathematical theory is omitted. However, when appropriate, sufficient references to applications and theoretical papers have been provided.

3 Conservative Finite Difference Method

The adopted numerical treatment is of paramount importance in what concerns the quality of the result. Since the governing equations are written in conservative form, it would seem reasonable that the numerical scheme should also possess this property. One approach widely used to numerically approximate (1) is the finite difference method. Great care must be taken when using finite difference to construct a finite difference as we need to ensure that the scheme is conservative. A finite difference scheme that is not conservative may propagate discontinuities at the wrong wave speed, if at all, giving inaccurate numerical results.

The computational domain is partitioned into a finite number of control volume $[x_{j-1/2}, x_{j+1/2}]$ around a nodal position x_i (see figure 2). The position of the right face $x_{j+1/2}$ is defined as $(x_j + x_{j+1})/2$. For each partition of the interval $[0, L]$, $x_{j+1/2}$ with $j = 1, \ldots, N$, we set $\Omega_J = [x_{j+1/2}, x_{j-1/2}]$.We multiplied the equation (1) by a weight function ϕ_h and the differential system is integrated over each control volume Ω_J to obtain the weak formulation which is the discrete equivalent of the conservation law.

$$\int_{\Omega_J} \partial_t U_h \phi_h \, dx - \int_{\Omega_J} F(U_h) \, \partial_x \phi_h \, dx + \Big[F(U_h) \, \phi_h\Big]_{j+1/2} - \Big[F(U_h) \, \phi_h\Big]_{j-1/2} = \int_{\Omega_J} S(U_h) \, \phi_h \, dx \tag{6}$$

If we assume a piecewise constant solution in each cell we can set

$$\phi_h = \begin{cases} 0 & \text{if} \quad U_h \notin \Omega_J \\ 1 & \text{if} \quad U_h \in \Omega_J \end{cases}$$

We then obtain a semi-discrete system of ordinary differential equations for the $U_j(t)$,

$$\frac{d}{dt} U = \frac{1}{\Omega_J} \mathcal{H}(U(t); j) \tag{7}$$

where the $\mathcal{H}$ is the spatial discretization operator. Problem that we consider in this report, the function U_h is discontinuous at the point $x_{j\pm1/2}$. Consequently the spatial operator $\mathcal{H}$ will depend on the two values of U_h at the point $x_{j\pm1/2}$.

If we replace the time derivative of (6) with a forward finite difference approximation in time,

$$U_i^{n+1} = U_i^n + \frac{dt}{dx} \Big[(F_{i+1/2} - F_{i-1/2})\Big] \tag{8}$$

where $F_{i\pm1/2}$ is called the numerical flux function. Finite differences written in the form of (7) are considered to be in *conservative form*, which simply states that the rate of change in $\mathbf{U}$ in a cell is equal to the difference in the fluxes entering the cell. It is the integral form of the governing equations which allows discontinuities in the solution. The specific difference between various schemes are the way in which they approximate the interface flux $F_{i+1/2} = F(U(x_{i+1/2}, t))$. Appropriate fluxes will be discussed later.

4 Upwind Scheme

Incorporating physics into numerical schemes ensure that we treat non-linear phenomena correctly. The Jacobian in the non-conservative form of the equations contains information that is needed if we want to incorporate physics in the discretized schemes. Since all eigenvalues are distinct and real (by definition of hyperbolic system), Jacobian matrix can be diagonalized to transform the SW system in a set of decoupled advection equations [1] for the characteristic variables.

Characteristic variables parameterize state variables (physics) in terms of the wave nature of the flow. In this characteristic space state variables can be viewed as point propagating along curves (path line) according

[1] this transformation is a similarity transformation

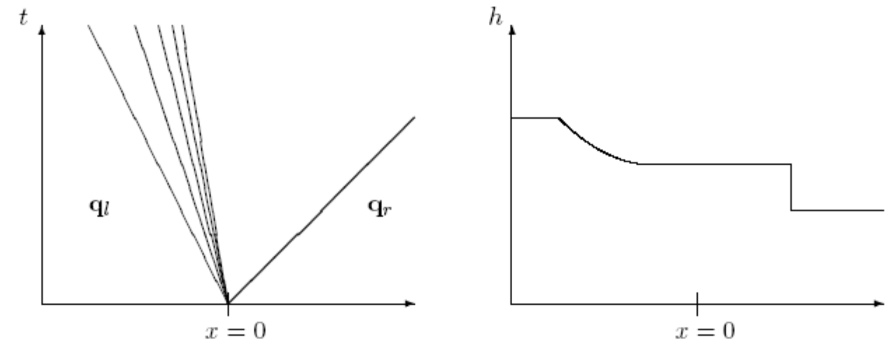

Figure 2: Riemann solution of the Shallow-water equations

to advection equation $v_t + \lambda\, v_x = 0$ with velocity equal to eigenvalues. These eigenvalues are sometimes called "characteristic speeds", their signs provide information about the directions of propagation, and curves in space-time upon which the information travels are sometimes called "characteristic curves". For a system of n equations there are n eigenvalues, and hence n families of characteristic curves.

One family of schemes which attempt to incorporate physics in numerical scheme is the so-called upwind scheme, or characterics based scheme since the space discretization use characteristics informations, guaranteeing that the propagation of information is consistent with the characteristics directions. In the case of supercritical flows this results in the correct direction of propagation of flow properties, both characteristic speeds are positive, indicating that information can only travel downstream. The introduction of physical properties in the discretization process can be done at two levels, namely the flux-vector splitting (FVS), considering only the sign of the eigenvalues, and at a higher level the Godunov-type scheme or Flux Difference Spitting (FDS), the space differencing depends on the sign of the eigenvalues and treats each non-linear wave according to its nature. The latter is based on the solution of the Riemann problem, making the shock an eigenvector of the system (through the Rankine-Hugoniot relation) propagating at the shock speed (eigenvalue).

4.1 Flux Vector Splitting

The FVS approach achieves upwinding by decomposing the flux vector into positive and negative components according to the sign of the eigenvalues of the Jacobian matrix. This method is attributed to Steger and Warming (1981), $\mathbf{F} = \mathbf{F}^+ + \mathbf{F}^-$. Then the governing equation becomes $\mathbf{U}_t + (\mathbf{F}^+)_x + (\mathbf{F}^-)_x = 0$. To explicitly consider the direction of flow, $(\mathbf{F}^+)_x$ is discretized using backward differencing and $(\mathbf{F}^-)_x$ is discretized using forward differencing. The identification of the upwind directions is done by $F^+ = (F + |F|)$ and $F^- = (F + |F|)$ as in usual way for upwind scheme. Different approach are used to evaluate $|F|$.

4.2 Godunov-type Scheme (Flux Difference Splitting)

The Godunov-type approach use either exact or approximate Riemann solutions between two adjacent cell to calculate the flux through the interface between them. The Riemann problem is a particular initial value problem (IVP) which consist of a conservation law or a system of conservation laws with a discontinuous initial solution.

In the traditional Godunov scheme, the numerical solution is considered piecewise constant in each mesh cell $[x_{j+1/2}, x_{xj-1/2}]$. The cell interface at $x_{j+1/2}$ separates two constant states $U^L = U_{i-1}$ and $U^R = U_i$

(connect a discontinuity) which form the Riemann problem.[2]. The numerical fluxes comes from evaluating the true flux function with solution from the Riemann problem at each cell interface

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - \frac{\Delta t}{\Delta x}\Big[\mathbf{F}(\mathbf{U}_{i+1/2}^L, \mathbf{U}_{i+1/2}^R)_{x/t=0} - \mathbf{F}(\mathbf{U}_{i-1/2}^L, \mathbf{U}_{i-1/2}^R)_{x/t=0}\Big] \tag{9}$$

where $\mathbf{U}_{i\pm 1/2}^R$ are the solutions of the Riemann problems at $x_{j\pm 1/2}$ given the initial data

$$U(x,0) = \begin{cases} U^L & x < x_{i-1/2} \\ U^R & x > x_{i-1/2} \end{cases}$$

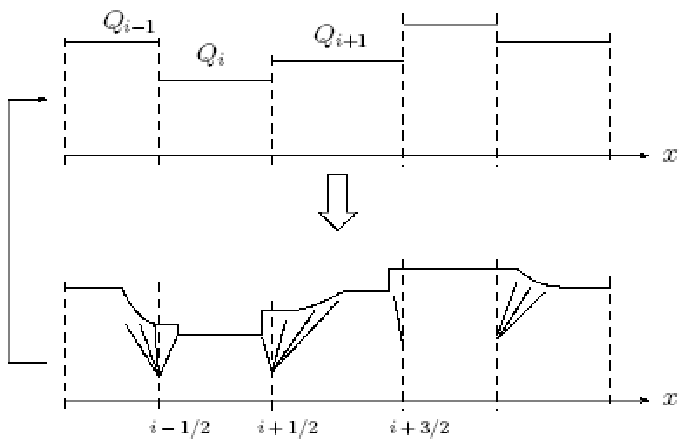

Figure 3: Godunov solution of the Riemann problem at the cell interface

Figure 3 is an example of a Riemann solution to the shallow water equations, with a shock in one characteristic family and a rarefaction in the other. The left figure show characteristic transitions, or waves emanating from the initial discontinuity in the $x-t$ plane. The right figure show the first component of the solution, the water depth h, at some time after $t=0$.

Riemann solvers in the context of the shallow water wave equations designate the class of schemes, [Godunov (1959), Roe (1981), Osher (1982) and van Leer (1982)], that may be viewed as providing exact or approximate solutions to a "local" model dam break problem (see figure 3).

4.3 MUSCL extrapolation

To reduce artificial dissipation while retaining non-oscillatory properties, it is necessary to resort to higher-order schemes equipped with flux or slope limiters (e.g Nujic 1995)).

Some form of reconstruction (usually polynomial) is then required to compute the interface values required by the flux computation.

In a MUSCL scheme, the piecewise scheme approximation is replaced by piecewise linear approximation, which give second-order spatial accuracy

$$\begin{aligned} \hat{U}_{i+1/2}^R &= U_{i+1} - \frac{1}{2}\Psi(r_{i+1}) \cdot \delta U_{i+1/2} \\ \hat{U}_{i+1/2}^L &= U_i \quad + \frac{1}{2}\Psi(r_i) \cdot \delta U_{i+1/2} \end{aligned} \tag{10}$$

[2]A full discussion of exact solutions to Riemann problems and how to construct such solutions is beyond the scope of this report, but van be found in Leveque

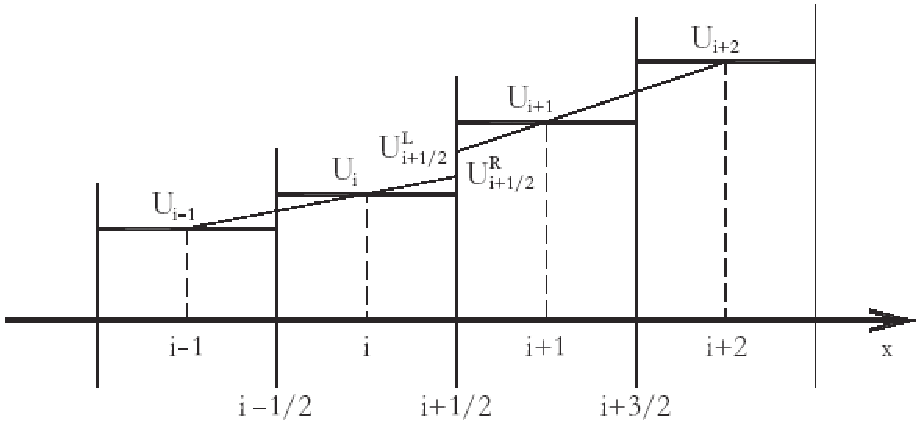

Figure 4: Reconstruction procedure

where $\delta U_{i+1/2} = U_{i+1} - U_i, \delta U_{i-1/2} = U_i - U_{i-1}, r_i = \delta U_{i+1/2}/\delta U_{i-1/2}, r_{i+1} = \delta U_{i+3/2}/\delta U_{i+1/2}$. Ψ is a *limiter* function. $\mathbf{r}$ is the ratio of successive gradients. The purpose of the limiter function Ψ is to limit the slope of the linear variations in order to avoid non-physical overshoots or undershoots in the numerical solution. The process of interpolating cell interface values from cell centre data is known as *reconstruction*. Various limiter functions can be found in the litterature e.g. the 'minmod', 'superbee' and 'van Leer' TVD limiter functions are well-known and widely used in the modern upwind schemes. A more comprehensive review of the methods can be found elsewhere (Hirsch (1988), Leveque (1990)).

4.4 Time Discretization

The semi-discretized set of equations (discretized in space but not in time) is a system of ODEs, one for every cell, that must be integrated in time by means of any of the methods available. In shock capturing models, explicit schemes rather than implicit schemes are typically used. Even though explicit schemes are subject to the usual Courant restriction (Hirsch 1988), there is a substantial reduction in the computing time required. The degrees of freedom can be written in the form

$$\frac{d}{dt} U_h = \mathcal{H}(U_h) \tag{11}$$

To obtain a method that is second-order accurate in time as well as space, we can discretize the ODE using a TVD Runge-Kutta[3] by performing a reconstruction in conjunction with a MUSCL.

$$\begin{aligned} \mathbf{u}_j^* &= \mathbf{u}_j^n - \frac{\Delta t}{\Delta x}\Big(\mathbf{f}_{j+1/2}^n - \mathbf{f}_{j-1/2}^n\Big) \\ \mathbf{u}_j^{n+1} &= 0.5\Big[\mathbf{u}_j^n + \mathbf{u}_j^* - \frac{\Delta t}{\Delta x}\Big(\mathbf{f}_{j+1/2}^* - \mathbf{f}_{j-1/2}^*\Big)\Big] - \Delta t\, g\, h\, S_0 \end{aligned} \tag{12}$$

4.5 Stability and Boundary Conditions

Due to the explicit discretization of the convective terms, a CFL number less than 1 is an upper limit for stability. The spatial step-size, Δx, is fixed and we use a variable time step,

$$\Delta t \leq \frac{\nu\, \Delta x}{\max_i(|\lambda_k|)} \tag{13}$$

[3]the two-stage Runge-Kutta (RK) method belongs to as family of RK methods which preserve TVD property under certain CFL conditions

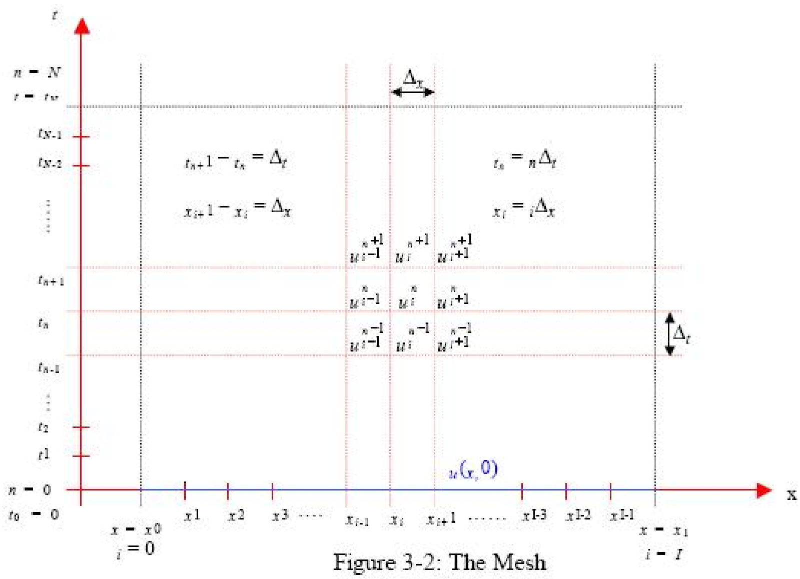

Figure 5: The One-Dimensional Mesh

where λ_k are the eigenvalues of the Jacobian matrix and ν is the required CFL number (Courant-Friedrihs-Levy). All schemes discussed in this report are stable for $\nu \leq 1$.

The numerical treatment of the boundary conditions is performed by setting the variables at the boundary edges based on the theory of characteristics (Alcrudo and Navarro, 1993). For subcritical flows, two external conditions must be specified at inflow boundaries, whereas only one is required at the outflow one.

5 Numerical Algorithms

In this section we present four schemes which have the properties discussed in the previous sections. We provide a short description for each of the scheme, papers are given in the bibliography for more details. As mentioned before, these schemes were chosen for their property to handle supercritical flow and discontinuity.

The first scheme has been developed by Nujic (1995), it's an adaptation of Shu and Osher scheme but in a simplified version. Use ENO (Essentially-Non Oscillatory) adaptative stencil idea technique on flux instead of state variables. The second scheme is again from Nujic paper, it is a simplified version of the Roe approximate solver. The third scheme is the Harten-Lax-Leer scheme. The last two schemes are based on approximate Riemann solvers which are another class of linear approximation methods, originally developed for the Euler equations of gas dynamics and extended to the homogeneous shallow-water equations. The fourth scheme is the Lax-Friedrichs central scheme. Is has been considered for comparison purpose and show that central scheme have problem to handle such a flow (see conclusion and future work section for more details).

5.1 Lax-Friedrichs Central Scheme

Since two of the three schemes that we are considering in this report are based on the Lax-Friedrichs scheme, it is worthwhile to present this scheme. It is first order and contains a diffusion term

$$u_j^{n+1} = 0.5\Big[u_{j-1}^n + u_{j+1}^n - \frac{\Delta t}{\Delta x}\Big(f(u_{j+1}^n) - f(u_{j-1}^n)\Big)\Big] \tag{14}$$

which is not in a conservative form. We can rewrite this in a conservative form *i.e.*

$$u_j^{n+1} = u_j^n - \frac{\Delta t}{\Delta x}\Big(F^*(u_j^n, u_{j+1}^n) - F^*(u_{j-1}^n, u_j^n)\Big) \tag{15}$$

where $F^*(u_j^n, u_{j+1}^n)$ and $F^*(u_{j-1}^n, u_j^n)$ represent the interface fluxes $F_{j\pm 1/2}$. Let's define the interface flux as

$$F^*(u_j^n, u_{j+1}^n) = \frac{\Delta t}{2\,\Delta x}\Big(u_j^n - u_{j+1}^n\Big) + \frac{1}{2}\Big[f(u_j^n) - f(u_{j+1}^n)\Big] \tag{16}$$

$$F^*(u_{j-1}^n, u_j^n) = \frac{\Delta t}{2\,\Delta x}\Big(u_{j-1}^n - u_j^n\Big) + \frac{1}{2}\Big[f(u_{j-1}^n) - f(u_j^n)\Big] \tag{17}$$

the u_j^n's in the F's will combine and cancel the leading u_j^n, and the $f(u_j^n)$ will cancel. So this F^* is the numerical flux for the LxF method. If the flow is constant so $u_j^n = u_{j+1}^n$ we have the consistency requirement $F(u, \ldots, u) = f(u)$.

5.2 Numerical Treatment of the Flux Term

In many practical applications, flows governed by the SWE are dominated by the source terms arising from the bed slope. This has had a profound influence on the numerical methods for flood propagation. The flux discretization must be performed in a way compatible with the bed slope contribution for otherwise the method will not be capable of passing the so called water at rest or lake at rest test case. This means that the numerical simulation of a mass of water initially at rest and enclosed in a reservoir with abrupt bottom will not hold the water at rest, but rather unphysical movement will be generated.

In order that the flux discretization be compatible with the bed slope source term Nujic's solution comprises first a splitting of the flux terms in a convective and a pressure contributions and then perform appropriate discretization. The splitting of the momentum flux is performed as $\mathbf{F} = f_c + f_p$

$$\mathbf{F} = \begin{pmatrix} hv \\ hv^2 + gh^2/2 \end{pmatrix} = \begin{pmatrix} hv \\ hv^2 \end{pmatrix} + \begin{pmatrix} 0 \\ gh^2/2 \end{pmatrix} \tag{18}$$

then the convective term is discretized by the scheme presented below and the pressure one is treated centrally leading to the following expression for the corresponding numerical flux at the cell interface $j+1/2$

$$\Big(\frac{gh^2}{2}\Big)_{j+1/2} = \frac{g}{2}\,(h_j^2 + h_{j+1}^2) \tag{19}$$

Further the bed slope term is computed normally at the centre of the corresponding cell, but with a spread of the slope discretization

$$\Big(gh\frac{\partial z}{\partial x}\Big)_j = \frac{g}{2}(h_{j+1} + h_{j-1})\Big(\frac{z_{j+1} + z_{j-1}}{2\,\Delta x}\Big) \tag{20}$$

where Δx represents the grid spacing. This treatment solves the water at rest problem and improves the robustness of the computation.

5.3 Numerical Scheme I: a simplified version of ENO schemes

The scheme presented here belongs to the flux-vector splitting upwind family discussed in the previous section. Use a local adaptive stencil to obtain information automatically from regions of smoothness when the solution develops discontinuities. As a result, approximations using these methods can obtain uniformly high-order accuracy right up to discontinuities, while keeping a sharp, essentially non-oscillatory shock transition

$$\mathbf{f}_{j+1/2} = \mathbf{f}^{+}_{j+1/2} + \mathbf{f}^{-}_{j+1/2} \tag{21}$$

This is a simplified version of the ENO scheme proposed by Shu and Osher (1988). The numerical flux is splitted up and each of the fluxes $f^{+}_{j+1/2}$ and $f^{-}_{j+1/2}$ is approximated up to the required order of accuracy using ENO moving stencil idea. Second-order accurate numerical fluxes are defined by:

$$\begin{aligned} f^{+}_{j+1/2} &= f^{+}_{j} + 0.5\delta f^{+}_{j} \\ f^{-}_{j+1/2} &= f^{-}_{j+1} - 0.5\delta f^{-}_{j+1} \end{aligned} \tag{22}$$

where

$$\begin{aligned} f^{+}_{j} &= 0.5(f_j + \alpha u_j), \\ f^{-}_{j} &= 0.5(f_j - \alpha u_j) \end{aligned}$$

where α represents some positive coefficients, and it is required that

$$\alpha \geq \max|\lambda_i|, \qquad i = 1, \ldots, m \tag{23}$$

5.4 Numerical Scheme II: a simplified version Roe's Approximate Solver

The numerical scheme is based on the first order Roe's scheme. Roe's approach replaces the Jacobian matrix in the divergent form by a constant Jacobian matrix (linearize about some constant state). The scheme read

$$f_{j+1/2} = 0.5\Big[F^R + F^L - |A_{j+1/2}|(U^R - U^L)\Big] \tag{24}$$

where $F^R = f(U^R)$ and $F^L = f(U^L)$. The matrix $A_{j+1/2}$ represents Roe's average matrix for the cell face $x_{j+1/2}$ and it satisfies $\Delta\, f_{j+1/2} = A_{j+1/2}\, \Delta u_{j+1/2}$, called the Rankine-Hugoniot condition.

In Nujic approach, the Roe's average matrix is replaced by a constant coefficient. To obtain a higher order of accuracy, the intermediate states U^L and U^R are obtained by using MUSCL type of extrapolation in an analog way as equations (9) and (10), using *minmod* function (limiter). The quantities δU_j and δU_{j+1} is approximated up to the required accuracy using a moving stencil ENO type (using a 3 points). The quantity δU_j is introduced to limit U_j to avoid the introduction of spurious oscillations in the numerical solution. Second-order accurate numerical fluxes are obtained for $f_{j+1/2}$ in the Lax-Friedrichs type scheme using

$$f_{j+1/2} = \frac{1}{2}(F^L + F^R) - \frac{\alpha}{2}(U^R - U^L) \tag{25}$$

The coefficient α is defined as

$$\alpha \geq \max_j|\lambda_j|, \qquad i = 1, 2, \ldots, N \tag{26}$$

where N is the number of computational nodes in the problem. This is no more than satisfying the Courant condition. The Lax-Friedrichs implementation of the ENO scheme is accurate, robust and very simple to implement.

5.5 Numerical Scheme III: HLL Approximate Riemann Solver

This scheme is based on estimating the speeds that information or waves propagate away from a Riemann problem. A linear solution with two discontinuities is then constructed[4], using estimates for the speeds of the

[4]Two speeds are used in the original HLL method even for equations with more than two characteristic families

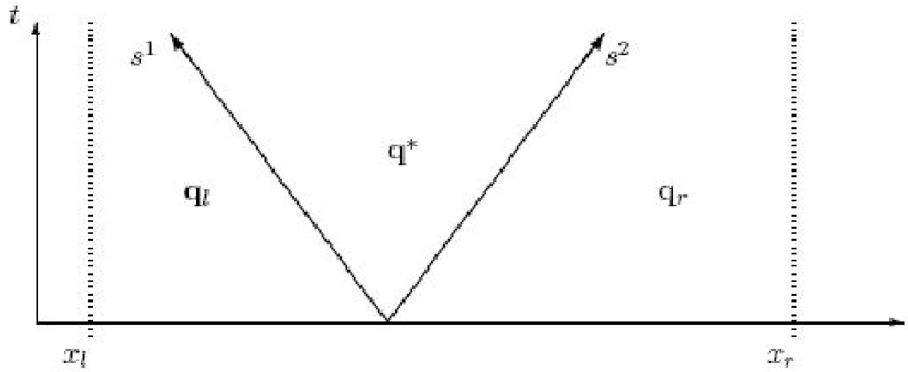

Figure 6: Constructing a solution using conservation and estimated speed

propagating discontinuities[5]. The first-order HLL scheme is based on the scheme described by Harten et *al.* (1983). The intercell flux, $f_{j+1/2}$ is evaluated by solving the approximate Riemann problem which involves only two shocks, although two rarefaction fans could be considered, separating three states; left-right, and intermediate state. For the Shallow-Water equations Toro(1992) presented a suitable HLL-type flux based on the suggested approximations of Harten. The intercell flux is chosen from

$$F^{HLL}(U^L, U^R) = \begin{cases} F^L & \text{if} \quad 0 \leq S_L \\ F^* & \text{if} \quad S_L \leq 0 \leq S_R \\ F^R & \text{if} \quad S_R \geq 0 \end{cases}$$

For the shallow-water wave equations, the intermediate states u^* and h^* are given by,

$$\sqrt{gh^*} = \frac{1}{2}(\sqrt{gh_L} + \sqrt{gh_R}) - \frac{1}{4}(u_R - u_L) \tag{27}$$

$$u^* = \frac{1}{2}(u_L + u_R) + \sqrt{gh_L} - \sqrt{gh_R} \tag{28}$$

and are used to estimate the shock speeds.

$$\begin{aligned} S_L &= \min\{u_L - \sqrt{gh_L}, u^* - \sqrt{gh^*}\} \\ S_R &= \min\{u_R + \sqrt{gh_R}, u^* + \sqrt{gh^*}\} \end{aligned} \tag{29}$$

which are estimates of the largest and smallest propagation speed. Although this may overestimate the true wave speed, Fraccarollo and Toro (1995) suggest that this enhance the stability of the scheme. These wave speeds are used to solve for the flux in the intermediate state

$$\mathbf{F}^* = \frac{S_R F_L - S_L F_R + S_L S_R (U_R - U_L)}{S_R - S_L} \tag{30}$$

by satisfying the integral form of the conservation law over a control volume. Note that the above given wave speeds are obtained under an assumption of wet bed, *i.e.* a non-zero flow depth h, on both side of the computational domain.

6 The Dam-Break Problem

The performance of a numerical scheme for the solution of the shallow water wave equations will be assessed by considering the Dam Break problem in a horizontal and frictionless channel. The 1D dam break is a very

[5]Contrast this to a method such as Roe solver, where a constant estimate to the Jacobian matrix is constructed first, and the eigenvalues of this estimate subsequently affect the approximate Riemann solution

useful benchmark test with a known exact solution. It provides extreme conditions to assess the numerical stability of the model. At time $t = 0$, the dam is instantly and totally removed. It creates a *bore* wave moving from left to right, and a *depression* wave, or *rarefaction* wave, propagating towards the left. To treat this problem numerically consider a fixed region $0 \leq x \leq 1$ with a barrier at $x = 0.5$. Initially, the water has a different depth on each side of the dam, h_L and h_R (Figure ?) and the water is assumed to be at rest. (The speed of the bore is the speed at which the discontinuity in the solution travels).

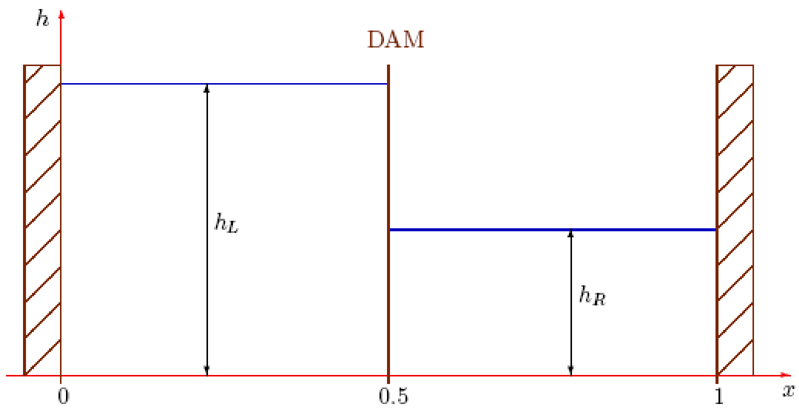

Figure 7: Initial conditions of the Dam break problem

In the test problem that we consider has no source term and the riverbed of constant depth:

$$u(x,0) = 0 \quad \text{and} \quad h(x,0) = \begin{cases} 1 & \text{if} \quad 0 \leq x \leq \frac{1}{2} \\ \phi_0 & \text{if} \quad \frac{1}{2} < x \leq 1 \end{cases}$$

which are illustrated in figure 7. The exact solution is given the ratio h_L/h_R determines whether the flow downstream of the barrier is subcritical or supercritical[6].

Note that, for this problem, if $\frac{1}{\phi_0} > 7.2$ then both eigenvalues $\mathbf{J}(\mathbf{U})$ are of the same sign and the downstream flow is supercritical. If $\frac{1}{\phi_0} < 7.2$ then eigenvalues $\mathbf{J}(\mathbf{U})$ are of the opposite sign and the downstream flow is subcritical. If the downstream flow is supercritical then difficulties can arise in accurately numerically approximating the wave speed of the discontinuity at $x = 0$. In our test case we use the value of $\phi_0 = 0.5$, which result in $\frac{1}{\phi_0} = \frac{1}{0.5} = 2 < 7.2$, hence the downstream flow is subcritical.

We obtain an exact solution of the dam break problem by using the analysis of Hudson(1999).

7 Numerical Results And Future Work

A number of numerical schemes, actually four, for solving the one-dimensional shallow-water wave equations applied to problems containing discontinuities in the solution have been examined. Results from several computations show that the overall performance of these algorithms can be considered as very good and allows for accurate open-channel flow computations involving hydraulic jump. In order to study the performance of the numerical scheme, simulations have been performed on a test case characterized by a supercritical flow (dam-break problem). A CFL number of 0.6 was chosen in order to compare the first and second order scheme introduced in section 6, respectively. Figures 8-11 present the results for the 1D dam

[6] A region of flow is said to be supercritical if both eigenvalues λ_1, λ_2 given by equations (7) are of the same sign, *i.e.* when $|u| > \sqrt{ch}$; otherwise the flow is said to be subcritical

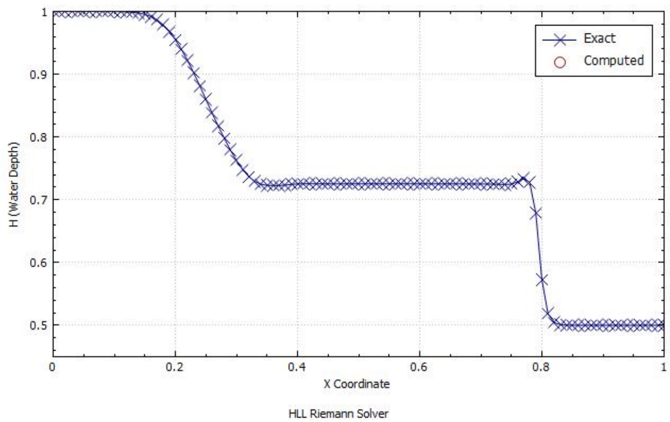

Figure 8: HLL Solver

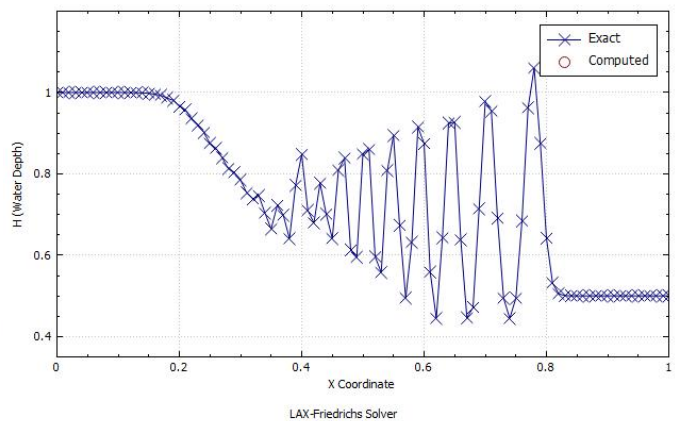

Figure 9: LAX-Friedrichs Solver

break with $h_L/h_R = 2.$, show ability of the model for simulating dam-break type flows. In the modeling, 100 computational cells are used, *i.e.* the spatial step is set to 0.1 m. The domain that we are considering is the following:

$$\Omega = \{(x,t) \in \mathbf{R}^2 \,|\, 0 < x < 1, \quad 0 < t < T\}$$

The numerical results compare very well with the exact solution, and show that the schemes are stable and that a little oscillations are introduced. HLL scheme show that the method can cope better with complex supercritical flows than other classical schemes. The algorithm is capable of capturing sharp discontinuities without generating spurious oscillations. With the exception of naive first-order schemes, all schemes produce reasonable results. First-order central scheme (e.g. Lax-Friedrichs) have a symmetry property about the eigenvalue (change sign) of the Jacobian, hence unable to distinguish between upstream and downstream influence (Hirsch 1988) which suffer of oscillations. Sometimes these oscillations can be minimized by using sufficiently small step-sizes.

Future work will include the incorporation of the following features:

- consider section of arbitrary width (rectangular)
- add the pressure term
- incorporate friction (Manning coefficient)
- different bottom topography, for example step in the bottom topography

This would be an intermediate step toward the real complex case. Our primary goal is to build a simulation environment where we could experiment different numerical simulation and also prototype in a simpler case (rectangular section). For this purpose, we are presently developing a set of computational tools that will help and facilitate our task of managing the change with respect to the numerical discretization (or schemes). Details of our computational framework will be available through a technical report (in preparation).

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