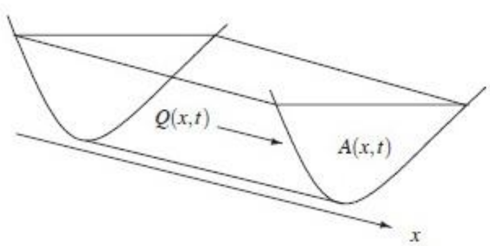
Introduction

Considerable work has been devoted to the numerical solution of the {*Shallow-Water Equations*} (SWE) not only for their inherent importance as regards modeling of many physical processes, ranging from river and channels flows to estuarine circulation and floods due to the dam or dike failure,
but also, for their mathematical difficulties, namely its non-linearity can give rise to discontinuous solutions currently referred to as bores or jumps. In fact, the SWE constitute a nonlinear system of partial differential equations (in one and two dimensions) of the hyperbolic type with a nonlinear source term.

Mathematical Model

Conservative Form of St-Venant Equations

One-dimensional open-channel flow is usually described in terms of water depth and discharge, and the evolution of these quantities is taken to be governed by the Saint-Venant equations, which simply express the conservation of mass and momentum along the flow direction.



St-Venant system can be written in different form, we consider the conservative form of the equation.

$$\partial_t \boldsymbol{U} + \partial_x \boldsymbol{F}(\boldsymbol{U}) = \boldsymbol{G} \qquad (1)$$

where

$$U = \begin{pmatrix} A \\ Q \end{pmatrix}$$

$$F = \begin{pmatrix} Q \\ Q^2/A + gI_1 \end{pmatrix}$$

$$G = \begin{pmatrix} 0 \\ gI_2 + gA(S_f - S_b) \end{pmatrix}$$

with F flux tensor and G source terms. Other variables are the following

- **U**: state vector of the state variables, respectively wetted area and discharge of section flow
- **F** : physical flux, convective and pressure
- S_f: friction term expressed by the Manning formula
- S_b: bottom slope term expressed as derivative of the bathymetry
- **A**: wetted cross-section area which depends **(x,h(x,t))**
- **g**: acceleration of gravity
- $\boldsymbol{I_1}$: pressure term given by $\int_0^{h(x)} (h(x) - \vartheta)\sigma(x,\vartheta)\, d\vartheta$
- $\boldsymbol{I_2}$: and given by $\int_0^{h(x)} (h(x) - \vartheta) \left[\frac{\partial\sigma(x,\vartheta)}{\partial x}\right] d\vartheta$ section width variation along x

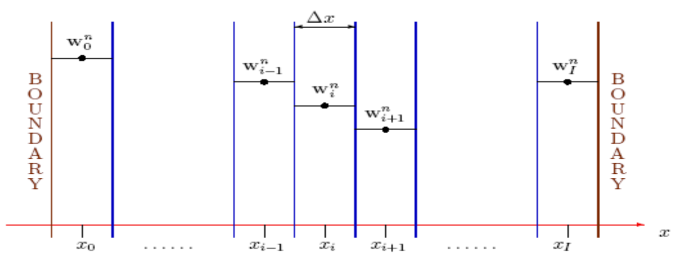
with h water depth, σ width for a fixed depth and ϑ depth integration variable along y axis.

Numerical Method

Finite Volume Discretization

A conservative finite difference method is used to solve these equations. The domain of the problem is represented by a collection of simple domains, called cell. In the 1-D case the domain [a,b] of the equation is discretized as a series of points x_i, i=0,...,N-1 where the solution "w" of the equation is discretized as. The problem consists of evaluating a discrete equation on each cell, the physical process is approximated by functions of desired type (polynomials or otherwise), and an algebraic equation relating physical quantities at selective points, called nodes, of the element are developed.

The discretized equations are constructed over each cell, these equations are in a conservative form, which mean that the variation of state variable inside a cell depend only on the fluxes (in/out).



Piecewise linear function in each cell

Time and Spatial Discretization

The framework contains basic building blocks for the numerical solution of the differential equation of ODE type:

$$U_t = L_\Delta(U;t) \qquad (2)$$

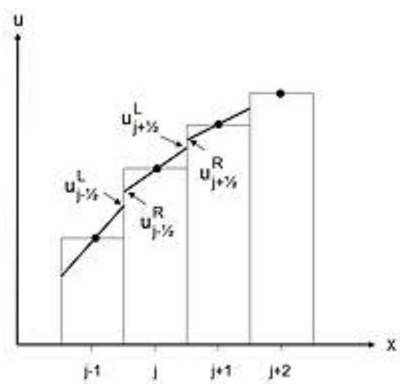
where L_Δ represents the differential operator in space i.e., one which does contain spatial derivative

$$L_\Delta = \partial_x F - \Delta t\, S - \Delta t\, P \qquad (3)$$

and U_t is the time derivative of the solution **U**. Writing equation in the above form allows us to implement separately the discretization of the differential operator L_Δ and the time scheme used for the evolution of the solution. This is called Method-of-lines (sometime semi-discrete).

MUSCL Reconstruction

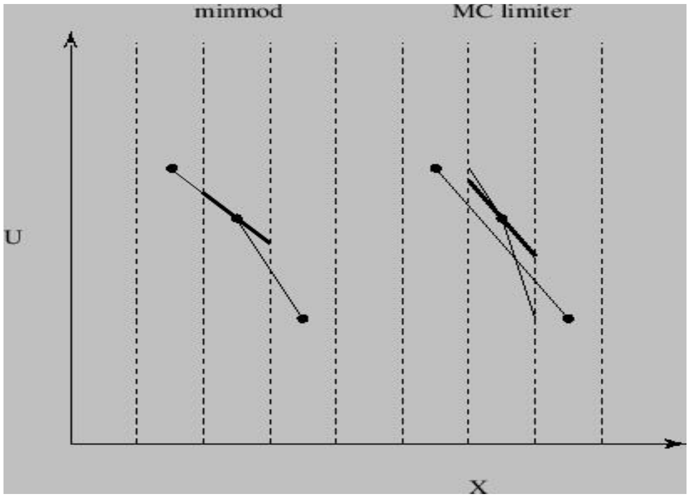
Below the graph show the process of reconstruction of the state variables at the interface.



Piecewise linear extrapolation

Slope Limiter

Since we are modeling mathematical function that have strong gradient (peak function), we need to use some technique to limit the slope of the function (called a slope limiter) thus to avoid spurious oscillations. There are many slopes limiter function, one of the most popular is the “*minmod*” function. It takes the minimum of adjacent gradient (of each cell)



Godunov-Type Scheme

The Godunov-type approach use either exact or approximate Riemann solutions between two adjacent cell to calculate the flux through the interface between them. The Riemann problem is a particular initial value problem (IVP), which consist of a conservation law or a system of conservation laws with a discontinuous initial solution.

Explicit Finite Difference Scheme (discretized equations)

Discrete representation of the numerical scheme that we used to solve the St-Venant equations. A conservative discretization is used and is expressed as follow:

$$U_i^{n+1} = U_i^n - \lambda \left\{ \mathrm{F}^{\mathrm{n}}_{\mathrm{i+1/2}} - \mathrm{F}^{\mathrm{n}}_{\mathrm{i-1/2}} \right\} - \Delta \mathrm{t}\, \mathrm{S_i}\ \Delta t\, P_i \quad (4)$$

where $F_{i\pm1/2}$is the numerical flux of the state variables **U** and **S** represent source terms (friction and bottom). Friction terms are evaluated according to the Manning formula. A finite volumes solver is used to integrate the simplified 1D Shallow-Water Equations (SWE) written in conservative form of unsteady open channel flow. This integration technique forms the basis of what is known as the finite volume method. The specific difference between various finite volume schemes is the way in which they approximate the interface convective flux $F_{i\pm1/2}$. This method can reproduce the discontinuities of a flow regime variations without incurring instabilities of the solution.